Bipartite Digraphs Debates

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Abstract

A novel graph-based model for aggregating dichotomous preferences is introduced. The output *opinion* is viewed as a consensual situation, paving the way of using graph operations to describe properties of the aggregators. The outputs are also dichotomous preferences which could be useful in some applications. New axiomatic characterizations of aggregators corresponding to usual majority or approval &disapproval rule are presented. Integrating and exploiting Dung's Argumentation Frameworks and their semantics into our model is another contribution of the present paper.

1 Introduction

The main objective of this paper is to introduce and study a new graph-based model for aggregating dichotomous preferences. Although dichotomous preferences over alternatives may lack the expressiveness to capture intensity of the preference, they are natural in many settings, and are studied in different approaches of decision making systems mainly related to approval voting (see [Brams and Fishburn, 1978; Laslier and Sanver, 2010; Vorsatz, 2007], among others) or in connection to randomized mechanisms ([Bogomolnaia and Moulin, 2004; Bogomolnaia *et al.*, 2005]).

Let us suggest a new possible application. In peer assessments systems (used in massive open online courses, or in evaluation of grant applications, see [Walsh, 2014; Alfaro and Shavlovsky, 2014]) the main objective is to get a fair grade for each agent based on the grades proposed by some other agents. Uniform grading is an obvious desideratum for the quality of the outcome. An approach to uniformize individual grading is to consider each grade given by an agent as good or bad, depending on how this grade compares to the average grade given by this agent.

In our dichotomous setting, we consider two disjoint nonempty finite sets $\mathbb{F}$ and $\mathbb{S}$. $\mathbb{F}$, referred as the set of *facts* (*issues*), is the set of alternatives in a decision making system, e.g. candidates in an election process, normative judgments, goods in purchasing systems, time slots in meeting scheduling systems, etc. Note that all facts are "positive", there are no negative facts. The *individuals* (*persons,agents*) in $\mathbb{S}$, the *society*, express their *opinions* (*dichotomous preferences*) that are pairs (L, DL) of disjoint subsets of $\mathbb{F}$: L is the set of facts on which the individual has a positive opinion and DL are the set of facts on which the individual has a negative opinion. The remaining facts in $\mathbb{F} - (L \cup DL)$ are indifferent or unknown to the individual. In applications in which $\mathbb{F}$ and $\mathbb{S}$ are not disjoint, we can consider disjoint copies of them.

Our focus is how to aggregate the individual opinions into a collective one. We are not only interested into positive collective position (as is intensively done in social choice theory) but also into the negative collective position (which can be used also as an explanation of the selected positive facts).

To illustrate our new model let us consider a simple mundane choice situation. Table 1 presents an example with available aliments, $F = \{a_1, a_2, a_3, a_4, a_5, a_6\} \subseteq \mathbb{F}$, and the preferences on these aliments of a group of five persons, $S = \{p_1, p_2, p_3, p_4, p_5\} \subseteq \mathbb{S}$, who want to have a common lunch.

	Liked	**Disliked**
$\mathbf{p_1}$	$\{a_1, a_3\}$	$\{a_5, a_4\}$
$\mathbf{p_2}$	$\{a_3, a_6\}$	$\{a_1, a_4\}$
$\mathbf{p_3}$	$\{a_3, a_5\}$	$\{a_1, a_4\}$
$\mathbf{p_4}$	$\{a_5, a_1, a_2\}$	$\{a_3\}$
$\mathbf{p_5}$	$\{a_5, a_1\}$	$\{a_3, a_2\}$
Majority	$\{a_3, a_1, a_5\}$	$\{a_4\}$

Table 1: Common Lunch Dillema.

As we can see, p_1 likes (agrees) a_1 and a_3 but dislikes (disagrees) a_5 and a_4. Similarly, we can read the opinions of the other members of S. Note that disliking an aliment may mean that the individual is allergic to it. The table is entitled *Common Lunch Dillema* since if we consider the *majority opinion* (include each fact in one of the two sets of liked and disliked facts using the majority rule) as output of the debate, then this has the unpleasant property that **each individual is allergic to an aliment in the collective output** $(\{a_3, a_1, a_5\}, \{a_4\})$! Note that this happens despite the majority rule gives (always) a consistent opinion, i.e. a disjoint pair of subsets of F.

The set of individual opinions listed in Table 1 (called "profile" in social choice theory) can be represented using a bipartite directed graph, which we call debate, as depicted in Figure 1. The two parts of a debate are the set of facts and the set of individuals. Each individual has as out-neighbors its set of liked facts and as in-neighbors the set of disliked facts. Since these two sets are disjoint, we have no symmetric pair of directed edges.

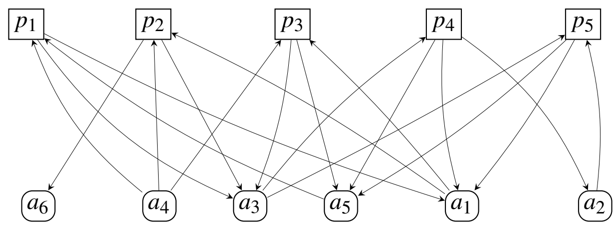


Figure 1: Common Lunch Dilemma – Bipartite Digraph Debate.

The main advantage of this model is its visual capacity and the symmetric treatment of the facts and individuals. Also well-established graph theoretical notions (digraph isomorphism, in-degree and out-degree of a vertex, induced sub-digraphs, digraph operations, etc.) can be used in describing normative conditions on the aggregating rules. The aggregate opinion (a disjoint pair of collective of agreed and disagreed facts) can also be viewed as a debate (bipartite digraph) in which each individual has the same sets of in-neighbors and out-neighbors (called consensual debate). In this way, axioms on the aggregating rules gain in expressivity (see Section 3).

To each digraph without loops we can associate a bipartite digraph, hence a debate. On the other hand, to each debate we can associate two conflict digraphs. These conflict digraphs are isomorphic to the digraph for which the debate is associated. This relationship is exploited in Section 4 to describe and analyze, from computational point of view, new (irresolute) aggregators corresponding to the digraphs representing Dung's argumentation framework. These give acceptable solutions to the Common Lunch debate discussed above.

The remainder of the paper is organized as follows. In the next Section, we introduce notations and basic definitions. In Section 3, aggregators are defined and characterizations of majority and approval&disapproval rules are presented. In Section 4, we propose an entire class of qualitative (irresolute) argumentative aggregators and discuss their properties. Finally, Section 5 concludes the paper.

2 Graph Based Framework

In this section we introduce our new model for aggregating dichotomous preferences and present graph theoretical concepts and notations used in the next sections. Recall from the Introduction the two disjoint finite non-empty sets $\mathbb{F}$ and $\mathbb{S}$ of *facts* and, respectively, *individuals*.

Definition 1 A *Debate* is a bipartite digraph $D = (F,S;E)$, where $\emptyset \neq F \subseteq \mathbb{F}$ and $\emptyset \neq S \subseteq \mathbb{S}$, and $E \subseteq F \times S \cup S \times F$ contains no symmetric pair of directed edges (i.e., at most one of the pairs (f,s) and (s,f) is a directed edge in E, for every $f \in F$ and $s \in S$).

Let $G = (V,E)$ be a digraph and $v \in V$ a vertex of G. The set of out-neighbors of v is denoted by v_G^+, that is $v_G^+ = \{u \in V|(v,u) \in E\}$. Similarly, the set of in-neighbours of v is $v_G^- = \{u \in V|(u,v) \in E\}$. These notations can be extended to set of vertices by considering, for every $S \subseteq V$, $S_G^+ = \cup_{v\in S} v_G^+$ and $S_G^- = \cup_{v\in S} v_G^-$ (clearly, $\emptyset_G^+ = \emptyset_G^- = \emptyset$).

If $D = (F,S;E)$ is a debate then, for every $s \in S$, s_D^+ is the set of **facts approved by the individual** s and s_D^- is the set of **facts disapproved by the individual** s. The pair (s_D^+, s_D^-) is referred as the **opinion** of individual s on the facts in F. By the above definition of a debate , $s_D^+ \cap s_D^- = \emptyset$. If $f \in s_D^+$ then s has a "positive" opinion on f, if $f \in s_D^-$ then s has a "negative" opinion on f, and if $f \notin s_D^+ \cup s_D^-$ then s has no opinion on f.

Let $D = (F,S;E)$ be a debate. If $F' \subseteq F$, the **sub-debate** induced by F' is the sub-digraph induced by $F' \cup S$ in D, and is denoted by $D^{F'}$. If $S' \subseteq S$, the **sub-debate** induced by S' is the sub-digraph induced by $F \cup S'$ in D, and is denoted by $D_{S'}$. For $s \in S$, the sub-debate $D_{S-\{s\}}$ is denoted by $D - s$.

If $D_i = (F_i,S_i;E_i)$ $(i = 1,2)$ are debates with $S_1 \cap S_2 = \emptyset$, then their **sum** is $D_1 + D_2 = (F_1 \cup F_2, S_1 \cup S_2; E_1 \cup E_2)$. Clearly, $E_1 \cup E_2$ does not contain symmetric edges, that is, $D_1 + D_2$ is a debate. With this notation, if $D = (F,S;E)$ is a debate such that $|S| \geq 2$ then, for every $s \in S$, we have $D = D_{S-\{s\}} + D_{\{s\}} = (D - s) + D_{\{s\}}$.

Two debates $D = (F,S;E)$ and $D' = (F',S';E')$ are **isomorphic** if there are bijections $\alpha : F \to F'$ and $\beta : S \to S'$ such that for all $f \in F$ and $s \in S'$ $(f,s) \in E$ if and only if $(\alpha(f),\beta(s)) \in E'$, and $(s,f) \in E$ if and only if $(\beta(s),\alpha(f)) \in E'$. Two isomorphic debates are denoted by $D \cong D'$ or $D \cong_{\alpha,\beta} D'$ (when we need to emphasize the isomorphism).

Definition 2 A debate $D = (F,S;E)$ is a *consensual debate* if there are $F_D^+, F_D^- \subseteq F$ such that $F_D^+ \cap F_D^- = \emptyset$ and, for every $s \in S$, we have $(s_D^+, s_D^-) = (F_D^+, F_D^-)$. $O_D = (F_D^+, F_D^-)$ is called the *common opinion* of D.

Since F_D^+ (F_D^-) are subsets of F, it is no danger to be considered as sets neighbors. Note that, in a consensual debate, each individual has the same opinion.

3 Aggregators

In this section we define opinions aggregation, describe our versions of well-known majority and approval&disapproval rules and prove their axiomatic characterization.

Definition 3 Let $\mathscr{D}(\mathbb{F},\mathbb{S})$ be the set of all debates $D = (F,S;E)$ with $F \subseteq \mathbb{F}$ and $S \subseteq \mathbb{S}$. An *aggregator* is a function $\mathbb{A} : \mathscr{D}(\mathbb{F},\mathbb{S}) \to \mathscr{D}(\mathbb{F},\mathbb{S})$ such that

- $\mathbb{A}(D)$ is a consensual debate for every $D \in \mathscr{D}(\mathbb{F},\mathbb{S})$, and
- if $D_1 \cong D_2$ then $\mathbb{A}(D_1) \cong \mathbb{A}(D_2)$.

In words, an aggregator is a functional $\mathbb{A}$ that maps each debate D into a consensual output debate $\mathbb{A}(D)$, in which every individual has the same opinion $O_{\mathbb{A}(D)} = (F_{\mathbb{A}(D)}^+, F_{\mathbb{A}(D)}^-)$. Also, $\mathbb{A}$ satisfies the usual social choice theory conditions of *neutrality* and *anonymity* (renaming the facts or the individuals does not change the output modulo this renaming).

We give now two examples of aggregators, corresponding to well-known social choice theory rules. In both examples, the first condition in the above definition is satisfied by construction and the second condition is satisfied since the output consensual debate depends only on the in-degree and out-degree of the fact vertices, and these are invariant under debate isomorphisms. After defining each of these aggregators we consider normative properties of them and prove that these offer novel interesting characterizations.

Majority rule. $\mathbb{A}_M : \mathscr{D}(\mathbb{F},\mathbb{S}) \to \mathscr{D}(\mathbb{F},\mathbb{S})$ such that $\forall D \in \mathscr{D}(\mathbb{F},\mathbb{S})$, $O_{\mathbb{A}_M(D)} = (F^+_{\mathbb{A}_M(D)}, F^-_{\mathbb{A}_M(D)})$, where

$$F^+_{\mathbb{A}_M(D)} = \{f \in F \mid |f^-_D| \geq \frac{|S|}{2}\},$$

$$F^-_{\mathbb{A}_M(D)} = \{f \in F \mid |f^+_D| > \frac{|S|}{2}\}.$$

Note that, by definition, a fact f is approved (disapproved) by the collective opinion if the number of individuals that like (dislike) f is at least (greater than) half of the total number of individuals. It is not difficult to see that $F^+_{\mathbb{A}_M(D)} \cap F^-_{\mathbb{A}_M(D)} = \emptyset$. Also, for every $f \in F$ we have $\mathbb{A}_M(D)^{\{f\}} = \mathbb{A}_M(D^{\{f\}})$, that is, the aggregate opinion on f depends only on the opinions of the individuals in the society on f: to find the aggregate opinion on f, we apply the aggregator on the debate $D^{\{f\}}$ obtained by considering the restriction of D to $\{f\}$ only. This is the usual social choice theory **Independence (I)** condition, that is, the aggregation is done **fact-wise**. In order to characterize the majority rule, we consider also the following conditions **Unanimity (U)**, **Cancellation (C)**, and **Faithfulness (F)**:

U If D is a consensual debate then $\mathbb{A}(D) = D$.

I For every debate $D = (F,S;E)$ and for every $f \in F$

$$\mathbb{A}(D)^{\{f\}} = \mathbb{A}(D^{\{f\}}).$$

C For every debate $D = (\{f\},S;E)$ with $|S| \geq 3$, if $s,p \in S$ are such that $f \in s^+_D \cap p^-_D \cup s^-_D \cap p^+_D$, then $\mathbb{A}(D) = \mathbb{A}(D_{S-\{s,p\}})$.

F If $D = (\{f\},\{s,p\};E)$ is a debate such that $f \in s^+_D \cap p^-_D \cup s^-_D \cap p^+_D$, then $O_{\mathbb{A}(D)} = (F^+_{\mathbb{A}}, F^-_{\mathbb{A}(D)}) = (\{f\},\emptyset)$.

In words, *Cancellation* says that in any debate over a single fact, if there are at least three individuals and two of them have contradictory opinions on this fact, then the output opinion is decided by the remaining individuals. *Faithfulness* says that the output opinion of a debate over a single fact with exactly two individuals with contradictory opinions has a positive position on this fact.

Theorem 1 *The Majority rule, $\mathbb{A}_M$, is the only aggregator $\mathbb{A}$ satisfying conditions* U, I, C, and *F*.

Proof. Obviously, $\mathbb{A}_M$ satisfies U, I, and F. To prove that $\mathbb{A}_M$ fulfills C, let $D = (\{f\},S;E)$ be a debate with $|S| \geq 3$, and $s,p \in S$ such that $f \in s^+_D \cap p^-_D$ (the proof is similar for $f \in s^-_D \cap p^+_D$). Since $f^-_{D_{S-\{p,s\}}} = f^-_D - \{s\}$, it follows that $|f^-_{D_{S-\{p,s\}}}| \geq \frac{|S|-2}{2}$ if and only if $|f^-_D| \geq \frac{|S|}{2}$. Similarly, since $f^+_{D_{S-\{p,s\}}} = f^+_D - \{p\}$, it follows that $|f^+_{D_{S-\{p,s\}}}| > \frac{|S|-2}{2}$ if and only if $|f^+_D| > \frac{|S|}{2}$. Hence $\mathbb{A}_M(D) = \mathbb{A}_M(D_{S-\{s,p\}})$.

Conversely, let $\mathbb{A}$ be an aggregator satisfying U,I,C, and F. We prove that $\mathbb{A}(D) = \mathbb{A}_M(D)$ for every debate $D = (F,S;E)$, by induction on $|S|$.

If $|S| = 1$, then D is consensual and by U we have $\mathbb{A}(D) = D$ and, since $\mathbb{A}_M$ satisfies U, $\mathbb{A}(D) = \mathbb{A}_M(D)$. Also, for $|S| = 2$, for every $f \in S$, $D^{\{f\}}$ is either consensual and $\mathbb{A}(D)^{\{f\}} = \mathbb{A}_M(D^{\{f\}})$ by U, or satisfies the hypothesis of F and again $\mathbb{A}(D)^{\{f\}} = \mathbb{A}_M(D^{\{f\}})$. By I, we have $\mathbb{A}(D) = \mathbb{A}_M(D)$.

In the inductive step, let $D = (F,S;E)$ be a debate with $|S| \geq 3$. By I, in order to prove that $\mathbb{A}(D) = \mathbb{A}_M(D)$ it is sufficient to prove that $\mathbb{A}(D)^{\{f\}} = \mathbb{A}_M(D^{\{f\}})$ for each $f \in F$. This follows either by U or by applying C and the induction hypothesis. □

Approval&Disapproval rule. $\mathbb{A}_{\text{A\&D}} : \mathscr{D}(\mathbb{F},\mathbb{S}) \to \mathscr{D}(\mathbb{F},\mathbb{S})$ such that $\forall D \in \mathscr{D}(\mathbb{F},\mathbb{S})$, $O_{\mathbb{A}_{\text{A\&D}}(D)} = (F^+_{\text{A\&D}}, F^-_{\text{A\&D}})$, where

$$F^+_{\text{A\&D}} = \{f \in F \mid |f^-_D| - |f^+_D| \geq |g^-_D| - |g^+_D|, \forall g \in F\},$$

$$F^-_{\text{A\&D}} = \{f \in F - F^+_{\text{A\&D}} \mid |f^+_D| - |f^-_D| \geq |g^+_D| - |g^-_D|, \forall g \in F - F^+_{\text{A\&D}}\}.$$

Let us consider the **score** of $f \in F$ as $score_D(f) = |f^-_D| - |f^+_D|$, that is the difference between the number of individuals, $|f^-_D|$, having a positive position on f, and the number of individuals, $|f^+_D|$, having a negative position on f. Hence, the facts maximizing this score are selected in the positive part of the aggregator's opinion. From the remaining facts, those having the minimum score are included in the negative part of the aggregator's opinion. Clearly, in this case, the aggregation is not fact-wise: despite computing the scores is done fact-wise, the decision of the aggregator on a fact depends on the scores obtained by the other facts. A characterization of Approval&Disapproval rule can be obtained by considering the above **Unanimity (U)** condition and the following two new conditions: **Summation (S)** and **Additivity (A)**.

S For every consensual debate $D_1 = (F_1,S_1;E_1)$ and for every debate $D_2 = (F_2,\{s\};E_2)$ with $s \notin S_1$, having $O_{D_1} = (F^+_1, F^-_1)$ and $O_{D_2} = (F^+_2, F^-_2)$, the aggregate debate of their sum, $\mathbb{A}(D_1 + D_2)$, is such that $O_{\mathbb{A}(D_1+D_2)} = (F^+, F^-)$, where

$$F^+ = \begin{cases} F^+_1 \cap F^+_2 & \text{if } F^+_1 \cap F^+_2 \neq \emptyset, \\ F^+_1 & \text{if } F^+_1 \cap (F^+_2 \cup F^-_2) = \emptyset \text{ and } |S_1| > 1, \\ F^+_1 \cup F^+_2 & \text{if } F^+_1 \cap (F^+_2 \cup F^-_2) = \emptyset \text{ and } |S_1| = 1, \end{cases}$$

and

$$F^- = \begin{cases} F^-_1 \cap F^-_2 & \text{if } F^-_1 \cap F^-_2 \neq \emptyset, \\ F^-_1 & \text{if } F^-_1 \cap (F^+_2 \cup F^-_2) = \emptyset \text{ and } |S_1| > 1, \\ F^-_1 \cup F^-_2 & \text{if } F^-_1 \cap (F^+_2 \cup F^-_2) = \emptyset \text{ and } |S_1| = 1. \end{cases}$$

A For every debate $D = (F,S;E)$ with $|S| \geq 2$, $\mathbb{A}(D) = \mathbb{A}\big(\mathbb{A}(D-s) + \mathbb{A}(D_{\{s\}})\big)$, for every $s \in S$.

In words, *Additivity* says that in any debate with at least two individuals the output consensual debate is the aggregate debate of the sum of the (consensual) sub-debate induced by any individual and the consensual aggregate debate of the debate obtained by deleting this individual. *Summation* shows how to obtain the aggregate debate of the sum between a consensual debate and a debate with a single individual.

Theorem 2 *The Approval&Disapproval rule, $\mathbb{A}_{A\&D}$, is the only aggregator $\mathbb{A}$ satisfying conditions* U, S, and *A*.

Proof. We show firstly that $\mathbb{A}_{\text{A\&D}}$ satisfies U, S, and A.

U If $D = (F,S;E)$ is a consensual debate with $O_D = (F^+,F^-)$, then

$$score_D(f) = \begin{cases} |S| & \text{if } f \in F^+, \\ -|S| & \text{if } f \in F^-, \\ 0 & \text{if } f \in F-(F^+ \cup F^-) \end{cases}.$$

Since $|S| \geq 1$, it follows that $F^+_{\text{A\&D}} = F^+$ and $F^-_{\text{A\&D}} = F^-$, that is, $\mathbb{A}_{\text{A\&D}}(D) = D$.

S Let $D_1 = (F_1,S_1;E_1)$ be a consensual debate with $O_{D_1} = (F_1^+,F_1^-)$, and $D_2 = (F_2,\{s\};E_2)$ be a debate with $s \notin S_1$ and $O_{D_2} = (F_2^+,F_2^-)$. Then, in the debate $D = D_1 + D_2$, the $score_D(f)$, for $f \in F_1 \cup F_2$, is:

$$score_D(f) = \begin{cases} |S_1|+1 & \text{if } f \in F_1^+ \cap F_2^+ \\ |S_1|-1 & \text{if } f \in F_1^+ \cap F_2^- \\ |S_1| & \text{if } f \in F_1^+ - (F_2^+ \cup F_2^-) \\ -|S_1|+1 & \text{if } f \in F_1^- \cap F_2^+ \\ -|S_1|-1 & \text{if } f \in F_1^- \cap F_2^- \\ -|S_1| & \text{if } f \in F_1^- - (F_2^+ \cup F_2^-) \\ 1 & \text{if } f \in (F_1-(F_1^+ \cup F_1^-)) \cap F_2^+ \\ -1 & \text{if } f \in (F_1-(F_1^+ \cup F_1^-)) \cap F_2^- \\ 0 & \text{if } f \notin F_1^+ \cup F_1^- \cup F_2^+ \cup F_2^-. \end{cases}$$

Now, it is easy to see that $F^+_{\text{A\&D}} = F^+$ and $F^-_{\text{A\&D}} = F^-$, where F^+ and F^- are defined in condition S.

A Let $D = (F,S;E)$ be a debate with $|S| \geq 2$.
For every $s \in S$ we have $D = (D-s) + D_{\{s\}}$. Using U and S it is not difficult to verify that $\mathbb{A}_{\text{A\&D}}(D) = \mathbb{A}_{\text{A\&D}}\big(\mathbb{A}_{\text{A\&D}}(D-s) + \mathbb{A}_{\text{A\&D}}(D_{\{s\}})\big)$.

Conversely, let $\mathbb{A}$ be an aggregator satisfying U,S, and A. We prove that $\mathbb{A}(D) = \mathbb{A}_{\text{A\&D}}(D)$ for every debate $D = (F,S;E)$, by induction on $|S|$. If $|S| = 1$, then D is consensual and by U we have $\mathbb{A}(D) = D$ and, since $\mathbb{A}_{\text{A\&D}}$ satisfies U, $\mathbb{A}(D) = \mathbb{A}_{\text{A\&D}}(D)$. In the inductive step, by S, $\mathbb{A}(D) = \mathbb{A}\big(\mathbb{A}(D-s) + \mathbb{A}(D_{\{s\}})\big)$ for $s \in S$. By the induction hypothesis and since $D_{\{s\}}$ is consensual we have
$\mathbb{A}\big(\mathbb{A}(D-s) + \mathbb{A}(D_{\{s\}})\big) = \mathbb{A}\big(\mathbb{A}_{\text{A\&D}}(D-s) + \mathbb{A}_{\text{A\&D}}(D_{\{s\}})\big)$.
Since $\mathbb{A}_{\text{A\&D}}$ satisfies S, $\mathbb{A}_{\text{A\&D}}(D-s) + \mathbb{A}_{\text{A\&D}}(D_{\{s\}}) = \mathbb{A}_{\text{A\&D}}(D)$ and since $\mathbb{A}$ satisfies U and $\mathbb{A}_{\text{A\&D}}(D)$ is consensual, we obtain $\mathbb{A}(D) = \mathbb{A}_{\text{A\&D}}(D)$. □

4 Argumentative (Irresolute) Aggregation

4.1 Dung's Theory of Argumentation

In this subsection we present the basic concepts used for defining classical semantics in abstract argumentation frameworks (AF) introduced by Dung in 1995, [Dung, 1995]. We consider U a fixed countable *universe* of arguments.

Definition 4 An *Argumentation Framework* is a digraph $AF = (A,E)$, where $A \subseteq U$ is a finite and nonempty set; the vertices in A are called *arguments*, and if $(a,b) \in E$ is a directed edge, then *argument a defeats (attacks) argument b*. A, the argument set of AF, is referred as $Arg(AF)$ and its attack set E is referred as $Def(AF)$. The set of all argumentation frameworks (over U) is denoted by $\mathbb{AF}$.

Two argumentation frameworks AF_1 and AF_2 are *isomorphic* (denoted $AF_1 \cong AF_2$) if there is a bijection $h : Arg(AF_1) \to Arg(AF_2)$ such that $(a,b) \in Def(AF_1)$ if and only if $(h(a),h(b)) \in Def(AF_2)$. h is called an *argumentation framework isomorphism*, and it is emphasized by the notation $AF_1 \cong_h AF_2$. If $S \subseteq Arg(AF_1)$ and h is an isomorphism between AF_1 and AF_2, then $h(S) \subseteq Arg(AF_2)$ is $h(S) = \{h(a) | a \in Arg(AF_1)\}$. Similarly, if $M \subseteq 2^{Arg(AF_1)}$, then $h(M) \subseteq 2^{Arg(AF_2)}$ is $h(M) = \{h(S) | S \in M\}$.

We define now the Dung's extension-based acceptability semantics as follows (see also [Baroni and Giacomin, 2007]).

Definition 5 An *extension-based acceptability semantics* is a function σ that assigns to every $AF \in \mathbb{AF}$ a set $\sigma(AF) \subseteq 2^{Arg(AF)}$, such that for every two argumentation frameworks $AF_1, AF_2 \in \mathbb{AF}$, if h is an isomorphism between AF_1 and AF_2 ($AF_1 \cong_h AF_2$) then $\sigma(AF_2) = h(\sigma(AF_1))$. A member $E \in \sigma(AF)$ is called a σ*-extension* in AF.

If $AF = (A,D)$ is an AF, σ a semantics and $a \in A$, then a is σ*-credulously accepted* if $a \in \bigcup_{S \in \sigma(AF)} S$ and a is σ*-sceptically accepted* if $a \in \bigcap_{S \in \sigma(AF)} S$.

The set S of arguments *defends* an argument $a \in A$ if $a^- \subseteq S^+$. The set of *all arguments defended by a set S* of arguments is denoted by $F(S)$. For $\mathbb{M} \subseteq 2^A$, $\mathbf{max}(\mathbb{M})$ denotes the set of maximal (w.r.t. set-inclusion) members of $\mathbb{M}$ and $\mathbf{min}(\mathbb{M})$ denotes the set of its minimal members. The main admissibility extension-based acceptability semantics are:

Definition 6 Let $AF = (A,E)$ be an AF.
- A *conflict-free set* in AF is a set $S \subseteq A$ with property $S \cap S^+ = \emptyset$ (i.e. there are no attacking arguments in S). We will denote $\mathbf{cf}(AF) = \{S \subseteq A | S \text{ is conflict-free set}\}$.
- An *admissible set* in AF is a set $S \in \mathbf{cf}(AF)$ with property $S^- \subseteq S^+$ (i.e. defends its elements). We will denote $\mathbf{adm}(AF) = \{S \subseteq A | S \text{ is admissible set}\}$.
- A *complete extension* in AF is a set $S \in \mathbf{cf}(AF)$ with property $S = F(S)$. We will denote $\mathbf{comp}(AF) = \{S \subseteq A | S \text{ is complete extension}\}$.
- A *preferred extension* in AF is a set $S \in \mathbf{max}(\mathbf{comp}(AF))$. $\mathbf{pref}(AF) := \mathbf{max}(\mathbf{comp}(AF))$.
- A *grounded extension* in AF is a set $S \in \mathbf{min}(\mathbf{comp}(AF))$. $\mathbf{gr}(AF) := \mathbf{min}(\mathbf{comp}(AF))$.
- A *stable extension* in AF is a set $S \in \mathbf{cf}(AF)$ with the property $S^+ = A - S$. We will denote $\mathbf{stab}(AF) = \{S \subseteq A | S \text{ is stable extension}\}$.

4.2 Debates vs Argumentation Frameworks

The conflicts between individual opinions in a debate can be naturally viewed as argumentation frameworks in order to use the collective acceptance of Dung's semantics in aggregation. On the other hand, every argumentation framework can be seen as a debate as explained below.

Definition 7 (i) If $D = (F,S;E)$ is a debate, then

its *facts argumentation framework* is f-$AF_D = (F,C)$, where $C \subseteq F \times F$ and $(f,g) \in F$ iff $f_D^- \cap g_D^+ \neq \emptyset$, and

its *opinions argumentation framework* is o-$AF_D = (S,C')$, where $C' \subseteq S \times S$ and $(s,t) \in C'$ iff $s_D^- \cap t_D^+ \neq \emptyset$.

(ii) If $AF = (A,E)$ is an AF without loops, then

the debate associated to AF is $D(AF) = (F_A, S_A; E')$, where $F_A = \{f_a | a \in A\}$ (f_a is the fact associated to a), $S_A = \{s_a | a \in A\}$ (s_a is the individual associated to a), and $E' = \{(s_a, f_a) | a \in A\} \cup \{(f_a, s_b) | (b,a) \in E\}$.

In words: (f,g) is an *attack* in f-$AF_D = (F,C)$ if there is an individual s which approves f and disapproves g; (s,t) is an *attack* in o-$AF_D = (S,C')$ if s disapproves a fact approved by t; the individual s_a agrees f_a in $D(AF)$ and disagrees all f_b for the arguments b attacked by a.

Note that $(f,g) \in F \times F$ is an attack in f-$AF_D = (F,C)$ if and only if the digraph $D + (f,g)$, obtained by adding the edge (f,g) to D, contains at least one $\vec{C}_3$. Similarly, $(s,t) \in S \times S$ is an attack in o-$AF_D = (S,C')$ if and only if the digraph $D + (s,t)$, obtained by adding the edge (s,t) to D, contains at least one $\vec{C}_3$.

These constructions are exemplified in Figure 2. The isomorphisms are not incidentally as the following theorem shows.

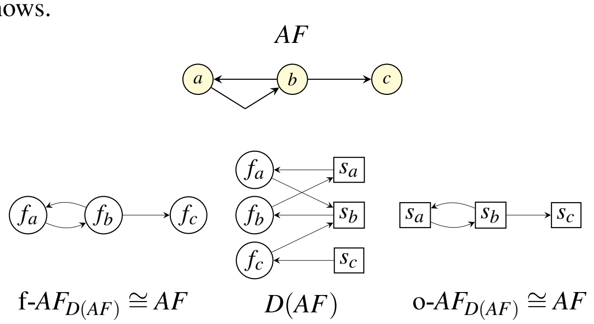


Figure 2: An AF AF, its associated debate $D(AF)$, with both (facts and opinions) AFs isomorphic to AF.

Theorem 3 *Let AF be an argumentation framework without loops and $D(AF)$ the debate associated to AF. Then the facts and opinions argumentation frameworks of the debate $D(AF)$ are isomorphic to AF:* f-$AF_{D(AF)} \cong AF \cong$ o-$AF_{D(AF)}$.

Proof. To prove that f-$AF_{D(AF)} \cong AF$, consider bijection $\varphi : F_A \to A$ given by $\varphi(f_a) = a$ for every $a \in A$. Then, $(f_a, f_b) \in Def($f-$AF_{D(AF)})$ if and only if there is $s \in S_A$ such that (s, f_a, f_b) is an induced $\vec{C}_3$ in $D(AF) + (f_a, f_b)$. By the definition of the debate $D(AF)$ it follows that $s = s_a$, and, since (f_b, s_a) is an edge in $D(AF)$, it follows that $(a,b) \in Def(AF)$. Conversely, if $(a,b) \in Def(AF)$ then, by the definition of the debate $D(AF)$, we have $(s_a, f_a) \in E(D(AF))$ and $(f_b, s_a) \in E(D(AF))$. Hence adding (f_a, f_b) to $D(AF)$ we obtain an induced $\vec{C}_3$, (s_a, f_a, f_b), in $D(AF) + (f_a, f_b)$. But this means that (f_a, f_b) is an attack in f-$AF_{D(AF)}$.

To prove that $AF \cong$ o-$AF_{D(AF)}$, consider bijection $\psi : S_A \to A$ given by $\psi(s_a) = a$ for every $a \in A$. Then, $(s_a, s_b) \in Def($o-$AF_{D(AF)})$ if and only if there is $f \in F_A$ such that (f, s_a, s_b) is an induced $\vec{C}_3$ in $D(AF) + (s_a, s_b)$. By the definition of the debate $D(AF)$ it follows that $f = f_b$, and, since (f_b, s_a) is an edge in $D(AF)$, it follows that $(a,b) \in Def(AF)$. Conversely, if $(a,b) \in Def(AF)$ then, by the definition of the debate $D(AF)$ it follows that $(s_b, f_b) \in E(D(AF))$ and $(f_b, s_a) \in E(D(AF))$. Hence adding (s_a, s_b) to $D(AF)$ we obtain an induced $\vec{C}_3$, (f_b, s_a, s_b), in $D(AF) + (s_a, s_b)$. But this means that (s_a, s_b) is an attack in o-$AF_{D(AF)}$. □

4.3 Irresolute Argumentative Aggregation

In this subsection we exploit the relationship between debates and argumentation frameworks developed in the above subsection in order to introduce new principles in doing debate aggregation.

More precisely we consider *aggregation correspondences*, $\mathbb{AC}$, which maps every debate $D \in \mathscr{D}(\mathbb{F}, \mathbb{S})$ into a set of consensual debates, $\mathbb{AC}(D)$, such that by specifying a rule of selecting a member of $\mathbb{AC}(D)$ we obtain an aggregator (that is, satisfying the second condition in Definition 3).

Let $D = (F,S;E) \in \mathscr{D}(\mathbb{F}, \mathbb{S})$ be a debate. An individual $s \in S$ is called a *strong eristic* if $s_D^- \cap t_D^+ \neq \emptyset$ for every $t \in S - \{s\}$. In words, an individual s is a strong eristic if it has a negative position on at least one of the facts agreed by every other individual t. An individual $s \in S$ is called a *weak eristic* if $s_D^- \cap t_D^+ \neq \emptyset$ for every $t \in S - \{s\}$ such that $t_D^- \cap s_D^+ \neq \emptyset$. In words, s is a weak eristic if it has a negative position on at least one of the facts agreed by every other individual t which has a negative position on a fact agreed by s. We can verify that p_4 is a strong eristic in the debate in Figure 1 and s_b is a strong eristic in the debate $D(AF)$ in Figure 2. Also, in this last debate, s_a is a weak eristic.

A *coalition* in $D = (F,S;E)$ is any subset $\mathscr{C} \subseteq S$ with $\mathscr{C} \neq \emptyset$. $\mathscr{C}$ is a *legal coalition* if the digraph obtained from D by contracting $\mathscr{C}$ is a debate, that is has no symmetric edges. Formally, $\mathscr{C}$ is a legal coalition if $D|\mathscr{C} = (F, (S - \mathscr{C}) \cup \{s_\mathscr{C}\}; E')$ is a debate, where $s_\mathscr{C}$ is a new "individual" ($s_\mathscr{C} \in \mathbb{S} - S$) and $(f,s) \in E'$ ($(s,f) \in E'$) if and only if $(f,s) \in E$ ($(s,f) \in E$) and $s \notin \mathscr{C}$ or $s = s_\mathscr{C}$ and there is $t \in \mathscr{C}$ such that $(f,t) \in E$ ($(t,f) \in E$). Hence a legal coalition is any non-empty set of individuals with the property that by merging them into a new single individual together with their incident edges, no symmetric edges are created. Note that $(s_\mathscr{C})^+_{D|\mathscr{C}} = \cup_{s \in \mathscr{C}} s_D^+$ and $(s_\mathscr{C})^-_{D|\mathscr{C}} = \cup_{s \in \mathscr{C}} s_D^-$. We can verify that $\{p_2, p_3\}$ is a legal coalition in the debate in Figure 1 and $\{s_a, s_c\}$ is a legal coalition in the debate $D(AF)$ in Figure 2.

Let $\mathscr{C}$ be a legal coalition in $D = (F,S;E)$. Then $\mathscr{C}$ is: an *oligarchy* if $s_\mathscr{C}$ is a strong eristic in $D|\mathscr{C}$; an *autarky* if $s_\mathscr{C}$ is a weak eristic in $D|\mathscr{C}$; a *strong autarky* if $s_\mathscr{C}$ is a weak eristic in $D|\mathscr{C}$ and for every $t \in S - \mathscr{C}$ such that $t^-_{D|\mathscr{C}} \cap (s_\mathscr{C})^+_{D|\mathscr{C}} = \emptyset$ there is $u \in S - \mathscr{C}$, $u \neq t$ such that $u^-_{D|\mathscr{C}} \cap t^+_{D|\mathscr{C}} \neq \emptyset$; a *maximal (minimal) strong autarky* if $\mathscr{C}$ is a strong autarky not strictly contained in (not strictly containing) a strong autarky. We can see that $\{p_2, p_3\}$ and $\{p_5\}$ are oligarchies in the debate in Figure 1, and $\{s_a, s_c\}$ and $\{s_b\}$ are oligarchies in the debate $D(AF)$ in Figure 2. These definitions are motivated by the next theorem that can be checked using Definitions 6 and 7.

Theorem 4 *A coalition $\mathscr{C}$ in $D = (F, S; E)$ is a legal coalition (respectively autarky, strong autarky, maximal strong autarky, minimal strong autarky, oligarchy) if and only if $\mathscr{C}$ is a conflict-free set (respectively admissible set, complete extension, preferred extension, grounded extension, stable extension) in the argumentation framework o-AF_D.*

By Theorem 3, for the debate $D(AF)$ associated to an argumentation framework AF, the above different type of coalitions translate to the corresponding admissible based extensions in AF. Hence the decision problems on argumentation frameworks can be polynomially transformed into instances on debates. Using the time complexity results on the corresponding decision problems for argumentation frameworks [Dunne and Bench-Capon, 2002] it follows that deciding if there is a maximal strong autarky with a positive position on a given fact in a given debate is an NP-complete problem and that deciding if a given fact belongs to the positive part of the opinions of every maximal strong autarky in a given debate is a Π_2^p-complete problem (see [Croitoru, 2014]).

Inspired by political practice, we consider a strategical way of coalition formation: some members of a coalition renounces at some liked facts in order to make the coalition legal. In this way the coalition has the same negative part of its merged opinion (which it is used to "attack" opinions of individuals not in coalition) despite of weakening the positive part. More precisely, if $\mathscr{C}$ is a coalition in $D = (F, S; E)$, a *$\mathscr{C}$-compromise* is the pair $(\mathscr{C}, E')$, where $E' \subset E$ is a set of edges (s, f) with $s \in \mathscr{C}$ such that in the debate $D' = D - E'$ (obtained from D by removing the edges from E') $s^+_{D'} \neq \emptyset$ and $\mathscr{C}$ is a legal coalition. $(\mathscr{C}, E')$ is called a σ-compromise if coalition $\mathscr{C}$ is σ in D', for $\sigma \in \{$ *autarky, strong autarky, minimal strong autarky, maximal strong autarky, oligarchy*$\}$. If $(\mathscr{C}, E')$ is a σ-compromise in $D = (F, S; E)$, then $D_{(\mathscr{C}, E')}$ is the consensual debate in which each individual opinion is the union of opinions of the members of $\mathscr{C}$ in $D - E'$. Note that in debates D with $s^+_D \neq \emptyset, \forall p \in S$, if $\mathscr{C}$ is σ then $\mathscr{C}$ is also a σ-compromise (by taking $E' = \emptyset$). Also, in the debates D with $|s^+_D| = 1, \forall p \in S$, a coalition $\mathscr{C}$ is σ-compromise if and only if $\mathscr{C}$ is σ. We define now our argumentative aggregation correspondences.

Definition 8 An *argumentative aggregation correspondence* is a function $\mathbb{AC}_\sigma$, which maps every debate $D \in \mathscr{D}(\mathbb{F}, \mathbb{S})$ into the following set of consensual debates

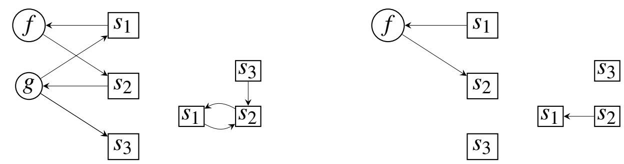
$$\mathbb{AC}_\sigma(D) = \{D_{(\mathscr{C}, E')} | (\mathscr{C}, E') \text{ is a } \sigma\text{-compromise}\}.$$

Example. Let us consider again the debate in Figure 1, and σ = oligarchy. We observed that $\mathscr{C}_1 = \{p_2, p_3\}$ and $\mathscr{C}_2 = \{p_5\}$ are oligarchies. Clearly $\mathscr{C}_3 = \{p_1, p_4\}$ is not legal. But, $(\mathscr{C}_3, E')$ with $E' = \{(p_1, a_3), (p_4, a_5)\}$ is an oligarchy-compromise. $F^+_{D_{(\mathscr{C}_3, E')}} = \{a_1, a_2\}$ and $F^-_{D_{(\mathscr{C}_3, E')}} = \{a_5, a_4, a_3\}$. Note that if we choose as output the merged opinion of an oligarchy (or oligarchy-compromise), each individual outside this coalition likes a fact which is disliked by some member of the oligarchy. Similar (argumentative) explanations can be made for other choices of σ, depending on the application for which the aggregation is considered.

Each argumentative aggregation correspondence $\mathbb{AC}_\sigma$ gives rise to an argumentative aggregator $\mathbb{A}_\sigma$, by specifying a rule to select one of the consensual debates in $\mathbb{AC}_\sigma(D)$. $\mathbb{A}_\sigma$ satisfies the second condition in Definition 3, by Theorem 4 and the invariance of the admissibility based semantics to the AFs isomorphism. Since the operators $\mathbb{A}_\sigma$ are not "fact wise" and are strongly dependent on the context of the debate to which they are applied, we have the following theorem.

Theorem 5 *Argumentative aggregation operators $\mathbb{A}_\sigma$ do not satisfy independence.*

Proof. Consider the debates D and $D^{\{f\}}$ in Figure 3 below.



D o-AF_D $D^{\{f\}}$ o-$AF_{D^{\{f\}}}$

Figure 3: Debates D and $D^{\{f\}}$ and their opinion AFs.

Since $|(s_i)^+_D| \leq 1$, a coalition $\mathscr{C}$ in these debates is σ-compromise if and only if it is σ. The only autarky in D is $\{s_1, s_3\}$ hence each individual opinion in $\mathbb{A}_\sigma(D)$ is $(\{f\}, \{g\})$. The only autarky in $D^{\{f\}}$ is $\{s_2, s_3\}$ hence each individual opinion in $\mathbb{A}_\sigma(D^{\{f\}})$ is $(\emptyset, \{f\})$. Therefore $\mathbb{A}_\sigma(D)^{\{f\}} \neq \mathbb{A}(D^{\{f\}})$. □

5 Discussion

The main contribution of this paper is to present a novel graph based framework for aggregating individual dichotomous preferences ("opinions") expressed as pairs of disjoint sets of positive and negative positions on a given finite set of "facts". These informations are represented as a bipartite digraph (with the two parts corresponding to the set of facts and the set of individuals) in which each individual's out-neighbors are its agreed facts, while the in-neighbors are its disagreed facts. This structure, called debate, is actually quite beautiful since we can formulate the aggregation as the task of associating to each debate a consensual one, in which each individual has the same in- and out-neighborhood. In this way, we have access to definable properties of aggregators in terms of simple graph operations. This is illustrated by our new axiomatizations of majority rule (a subject started by [May, 1952] and followed by several papers, e.g. [Maskin, 1995], [Woeginger, 2003], [Miroiu, 2004], etc.) or of approval&disapproval rule (a subject starting with [Brams and Fishburn, 1978], followed by several papers, see [Xu, 2010]) which is different from that given in [Alcantud and Laruelle, 2013]. The rich combinatorial structure of debates is exploited by the use of two argumentation frameworks associated, giving the possibility of devising qualitative aggregators in which the output can be "justified" in terms of argumentation semantics. The framework has an important intuitive and visual flavor, being simple and effective as the proof of Theorem 5 shows. As future work, we intend to provide the parts F, S with combinatorial/ logical structure such that our bipartite model extends to other social choice theory [Arrow, 1963] and Judgment Aggregation [Lang *et al.*, 2011] subjects.

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